## The statement

**Theorem 1.** *Let $f_i \in \mathbb{R}[x_1, \cdots, x_n]$ $i \in [m]$. Then the number of connected components of the locus defined by $f_i \geq 0$ is bounded by $\frac{1}{2}(D+2)(D+1)^{n-1}$ where $D = \sum_i \deg f_i$*

## An important lemma

**Lemma 2.** *Number of non-degenerate solutions of $f_1 = f_2 = \cdots f_m = 0$ is at most $\prod_i \deg f_i$*

*Proof.* Important thing is that since $\mathbb{R}$ is not algebraically closed we can't use Bezout's theorem directly. Let $a$ be a non-degenerate solution. By definition its Jacobian is non-singular and viewing $f_i$ now as elements of $\mathbb{C}[\bar{x}]$, the inverse function theorem says that $a$ is an isolated root. Now, we may apply Bezout's theorem and conclude that the number of such $a$'s is $\leq \prod_i \deg f_i$ □

## Milnor-Thom for a hypersurface

Before we do this there is an important technical proposition we need.

**Proposition 3.** *If $f$ is such that $\nabla(f)(x) \neq 0$ $\forall x \in V(f)$, then the system of equations, $f = \partial_1 f = \cdots \partial_{n-1} f = 0$ has only non-degenerate solutions.*

*Proof.* Direct from [Burgisser et al., 2010]. Let $V := Z(f, \partial_1 f, \cdots, \partial_{n-1} f)$. Define $g : V \to S^{n-1}$ as $g(x) = \frac{\nabla f}{||\nabla f||}(x)$. Since the graph of $g$ can be realized as a semi-algebraic set defined by

$$\{ \ y_i \partial_i f(x) \geq 0, \ y_i^2 ||\nabla f(x)||^2 = \partial_i f(x)^2 \mid i \ \in \ [n] \ \}$$

By the semi-algebraic Morse-Sard theorem we have that the space of the critical values has dimension $< n$. Thus, $\exists w(\text{ say }) = (0, \cdots, 0, 1)$ such that $w, -w$ are not critical values. Now, $g^{-1}(w) \cup g^{-1}(-w) = V$ as the gradient is non-zero but the first $n-1$ partial derivatives are. Let $\alpha \in V$. We need that $\alpha$ is non-degenerate. For any $x \in \mathbb{R}^n$, denote by $x'$ its projection to first $n-1$ coordinates. Since, $n^{th}$ derivative is non-zero we can use implicit function theorem to obtain a $C^\infty$ function $h$ such that the map $x' \to (x', h(x'))$ is a diffeomorphism to a neighbourhood around $\alpha$. Now,computing partial derivatives we obtain,

$$\begin{aligned}
\partial_i f(x', h(x')) &= -\partial_n f(x', h(x')) \partial_i h(x') \ \ i < n \\
\implies \partial_i h(\alpha') &= 0 \ \ i < n \\
\partial_i g_j(\alpha) &= -\partial_{i,j}^2 h(\alpha') \ \ i, j < n \\
\implies \partial_{i,j}^2 f(\alpha) &= -\partial_n f(\alpha) \partial_{i,j}^2 h(\alpha') \ \ i, j < n
\end{aligned}$$

□

**Theorem 4.** *Let $f \in \mathbb{R}[x_1, \cdots, x_n]$ be such that $n \geq 2, \deg f \geq 2, \nabla f(x) \neq 0$ $\forall x \in V(f)$ and $V(f)$ is compact, then, $b_0(V(f)) \leq \frac{1}{2} d(d-1)^{n-1}$ where $d := \deg f$*

*Proof.* Let $V(f) = \bigcup_i V_i$ where $V_i$ are the connected components. Since closed subsets of compact sets are compact, $V_i$ is compact. Since, $V_i$ is a connected component, it is irreducible and is therefore a variety. From the definition of dimension of a variety (See Mumford [1976]), since every point is smooth, its dimension is $n-1$. Define $\pi_n := (x_1, \cdots, x_n) \to x_n$. Let $p_i$ be the minimum of $\pi_n$ on $V_i$ and let $q_i$ be the maximum. At both points the gradient should be along the derivative of $\pi_n = (0,, \cdots, 0, 1)$. Thus, each such point, $p_i, q_i \in V(f, \partial_1 f, \cdots, \partial_{n-1} f)$. Moreover, $\forall i\ p_i \neq q_i$ because if not, then that implies that $\forall v \in V_i,\ \pi_n(v) = \pi_n(p_i) =: a$ Now, that means that $V_i \subset V(x_n - a)$. But we know from above that $\dim V_i = n-1$. Thus, $V_i = V(x_n - a)$ which contradicts compactness of $V_i$. Now, $b_0(V(f)) = \frac{1}{2}|\{p_1, q_i\}| \leq \frac{1}{2}|V(f, \partial_1 f, \cdots, \partial_{n-1} f)|$. By the above proposition all the zeroes are non-degenerate and by Lemma 2 we get the required bound. □

## Extending to semi algebraic sets defined by many polynomials

Let $S = \{\bar{x} \mid f_1(\bar{x}) \geq 0, \cdots, f_m \geq 0\}$. Dealing with this presents us with 2 issues that prevents us from using the previous machinery - One is that is not a zeroset, and the other is it's not necessarily compact. We solve the second issue first.

**Solving non-compactness** - Since we have a metric namely the Euclidean one on $\mathbb{R}^n$ we simply look at $S \cap B_r$ i.e. those points in S with distance from origin at most r. This can be realized by adding another polynomial constraint $f_0^r = r^2 - (\sum_i^n x_i^2) \geq 0$. The following lemma shows that obtaining a bound for this restriction suffices.

**Lemma 5.** *Let $K_i \subset \mathbb{R}^n\ \forall i \in \mathbb{N}$ such that $K_i \subset K_{i+1}$, then $b_0(\cup_{i\in\mathbb{N}} K_i) \leq sup_{i\in\mathbb{N}}\, b_0(K_i)$*

*Proof.* Let $C_1, \cdots, C_s$ be the connected components of $\cup_{i\in\mathbb{N}} K_i$. This implies that $\exists k_j\ C_j \cap K_t \neq \phi\ \forall t \geq k_j$. Choose, $m = max_j\ k_j$. Now for each $K_m$ and beyond, the intersection with each $C_i$ is non-trivial. Moreover, they can't merge into the same connected component. Thus, $sup_{i\in\mathbb{N}}\, b_0(K_i) \geq b_0(K_m) \geq s = b_0(\cup_{i\in\mathbb{N}} K_i)$ □

Applying this lemma with $K_n = S \cap B_n$ will give us that we need to just upper bound the compact set $S \cap B_n$ for an arbitrary (but fixed) n.

**Making it a zeroset** -To do this we modify $S \cap B_r$ by adding an $\epsilon$ to each $f_i$ and adding the polynomial constraint $f_{n+1} = \prod_i (f_i + \epsilon) \geq \delta\ \ , \epsilon \geq \epsilon^{m+1} \geq \delta > 0$. Thus to clarify $S_{r,\epsilon,\delta} := \{\bar{x} \mid f_0^r + \epsilon \geq 0, \cdots, f_n + \epsilon \geq 0, f_{n+1} \geq \delta\}$. Let's look at the boundary of $S_{r,\epsilon,\delta} := \partial S$. At the boundary at least one of the inequalities should be tight and the point be in S. But if any except the last is 0, the last inequality can't hold. Thus the boundary is defined by $\partial S = V(f_{n+1})$. This is clearly compact. we can make it non-singular by choosing $\delta$ appropriately (This is by Sard's theorem as we need to choose a $\delta$ such that it isn't a critical value of $f_{n+1}$ and that is possible as this set isn't dense). Applying Theorem 4 we get that.

$$b_0(\partial S) \leq \frac{1}{2}(D+2)(D+1)^{n-1}\ \ ,\ \ D = \sum_{i=1}^{m} \deg f_i$$

To make this exercise meaningful, we need the following result.

**Lemma 6.** $b_0(S) \leq b_0(\partial S)$

*Proof.* Since connected components are disjoint if they meet at boundary they do so in different components. Thus, we are done if we show that each connected component C of S satisfies $C \cup \partial S \neq \phi$. For a contradiction assume C doesn't. Then, for each $x \in C$, we have $B_{r_x} \in S \setminus \partial S$. The union of these is an open cover of C. Since, S in compact so is S and thus we have a finite subcover. Let R be the min radius $r_x$ of this set of this finite set of balls. The open set $\{x \mid dist(x, C) < R\} \in S \setminus \partial S$ is $\supset C$ and is clearly connected. This is a contradiction as a connected component is the maximal such. □

So we have that, $b_0(S_{r,\epsilon,\delta}) \leq \frac{1}{2}(D+2)(D+1)^{n-1}$. $S_r = \bigcap_{\epsilon<1} S_{r,\epsilon,\epsilon^{m+1}}$ To wrap up things we just need the other counterpart of 5 and that we mention without proof.

**Lemma 7.** *Let $K_i \subset \mathbb{R}^n$ $\forall i \in \mathbb{N}$ such that $K_i \supset K_{i+1}$, then $b_0(\cap_{i\in\mathbb{N}} K_i) \leq lim_{i\to\infty}\, inf\, b_0(K_i)$*

## Blackboxes

**Theorem 8** (Bezout's Inequality)**.** *The number of isolated solutions of $f_1 = f_2 = \cdots f_n = 0$ are atmost $\prod_i \deg f_i$ where $f \in k[\bar{X}]$ such that k is algebraically closed.*

**Theorem 9** (Semi-algebraic Morse-Sard)**.** *Direct from Burgisser et al. [2010]. Let $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^m$ be semi-algebraic subsets and smooth submanifolds, and let $\phi : V \to W$ be a smooth, semi-algebriac map. Let*

$$\Sigma := \{a \in V \mid rk\, d_a\phi < W\}$$

*denote the set of critical points of $\phi$. Then dim $\phi(\Sigma)$ is $<$ dim W*

**Theorem 10** (Implicit function theorem)**.** *Let f be a $C^k$ map $U \subset \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n$. Let $p = (x, y) \in U$ such that the derivative $Df(p)$ restricted to first n coordinates is invertible. Then there is a neighborhood $V \times W$ of p and a $C^k$ smooth map $h : W \to V$ such that $x = h(y)$ and $f(h(y), y) = 0$*

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The above exposition follows basically the approach of Milnor [1964] but uses the elementary (avoiding Cech cohomology) proofs of certain results as in Burgisser et al. [2010] to prove the required result.

## References

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