An Elementary Route to Grassmannians

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Certificate

This is to certify that Tushant Mittal (MATS654) has undergone training at Indian Institute of Science Education and Research for a period of 8 weeks from 17^{th} May to 21^{st} July, 2016, under my guidance. His performance has been satisfactory so as to fulfill all the requirements for successful completion of the program. This project report titled "An Elementary Route to Grassmannians " is a bona fide testimony of the work carried out by him .

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1. Introduction

1.1 My journey

Algebraic Geometry rests on the foundation of *Commutative Algebra* and I started the project by grasping its key ideas.Since the field of Algebraic Geometry is very diverse, it was confusing to decide what to study and I tried to dabble in a few of them. I read about *Gröbner Basis* which is the main object of interest in *Computational Algebraic Geometry* approach and very useful tool in algorithm design. I read about unification of conics which motivated me to look up *Projective Geometry*. The question of existence of a rational parametrisation of a curve seemed interesting and I worked it out for a circle but did not follow it up and would like to do it in the future. Next in line was the group law on cubics which is a fascinating topic and serves as a great introduction to the theory of elliptic curves. Reading up the general *Bezout's Theorem* (I read up one easier case) and *Reimann-Roch* would be one my next goals so as to complete the proof of associativity of the addition on cubics. I was then introduced to the problem of *Schläfli's Double Six* which comes under the branch of *Enumerative Algebraic Geometry*and I finally learnt about Grassmanians.

1.2 Abstract

We are all very familiar with the idea of representing geometric entities as solutions of a set of polynomial equation.Thus we have equations defining a line,circle etc...But what about the set of all lines satisfying some given property or set of all planes? Can we write down equations for them ? This is the central theme of the report.We first explore an example problem and then move on slowly to prove that the Grassmannian is a variety.The proof given is very elementary and a reader with a knowledge of basic linear algebra should be able to follow it.This is a departure from the standard proofs found in most of the texts which uses exterior algebra which although makes it neat requires some prior exposure which limits its accessibility.Theory that I have read from textbooks is not replicated here to avoid unnecessary repetition and only the parts that I have worked out on my own have been presented.

2. Hyperboloids

2.1 Locus of lines intersecting 3 skew lines

Let L_1, L_2, L_3 be 3 mutually skew lines. The goal is to find the set of lines which intersect all 3 of them.Without loss of generality we can assume,

$$\begin{aligned} L_1 &= (x, 0, 0) \\ L_2 &= (y, my, k), k \neq 0 \\ L_3 &= (a + \lambda v_1, b + \lambda v_2, c + \lambda v_3) \end{aligned}$$

We take the third line to be a general one with the conditions that,

$$\begin{gathered} \frac{b}{c} \neq \frac{v_1}{v_2} \\ (k - c) \neq v_3(am - b) \end{gathered}$$

Now let L be a line that intersects all the 3 lines.
Say it intersects L_1 at $(x_0, 0, 0)$ and L_2 at (y_0, my_0, k)

$$\Rightarrow L = (x_0 + t(y_0 - x_0), tmy_0, tk)$$

Since, L intersects L_3, $\exists t, \lambda$ such that,

$$(a + \lambda v_1, b + \lambda v_2, c + \lambda v_3) = (x_0 + t(y_0 - x_0), tmy_0, tk)$$

Solving this for y_0 after eliminating t and λ, we get

$$y_0 = \frac{bkv_1 - akv_2 - ((c-k)v_2 - bv_3)x_0}{cmv_1 + mv_3x_0 - cv_2 - (am - b)v_3}$$

From the line equation we can deduce the following relations,

$$x_0 = \frac{kmx - ky}{mk - mz}$$
$$y_0 = \frac{ky}{mz}$$

Replacing these in the solution for y_0 we get a 2 degree curve which contains all the lines which intersect the 3 lines.Let us take an example (the "general" case just looks uglier without having any extra generality).
Take $k = m = b = v_3 = 1, a = c = v_1 = v_2 = 0$. We get,

$$x_0 = \frac{x - y}{1 - z}, y_0 = \frac{y}{z} \;\; \& y_0 = \frac{x_0}{1 + x_0} = 1 - \frac{1}{1 + x_0}$$
$$\Rightarrow \frac{y}{z} = 1 - \frac{1}{1 + \frac{x-y}{1-z}} \Rightarrow \frac{1}{1 - \frac{y}{z}} = 1 + \frac{x - y}{1 - z}$$

The final equation is,

$$y(1 - z) = (x - y)(z - y)$$

This is the equation of a hyperboloid and it looks like this [Dev16],

Figure 2.1: $y(1 - z) = (x - y)(z - y)$

One interesting thing to note is that the 3 skew lines also lie on this hyperboloid. This shows that the hyperboloid contains not just the lines that intersect them but also some other lines and this leads us to ask what these other lines are.

2.2 All the lines on a hyperboloid

Let $x^2+y^2=1+z^2$ be the equation of a hyperboloid.
Take any point $P=(x_0,y_0,z_0)$ on it.

Theorem 1. *There exists 2 distinct lines passing through P*

Proof. Let L pass through $Q=(a,b,c)$ on the hyperbola.

If $L=(x_0+t(a-x_0),y_0+t(b-y_0),c-tz_0)$ lies on the hyperbola,

$$\begin{aligned}(x_0+t(a-x_0))^2+(y_0+t(b-y_0))^2 &= 1+(c-tz_0)^2\ \forall\, t\in\mathbb{R}\\ \Rightarrow (t^2-t)(ax_0+by_0-cz_0-1) &= 0\\ ax_0+by_0-cz_0 &= 1\end{aligned}$$

This is the equation of a plane. Therefore Q lies on the points of intersection of the plane and the hyperboloid. Since,the plane equation is linear eliminating one of the variables and substituting it in the hyperboloid equation will give us the equation of a conic passing through P and Q such that every point on the line through PQ lies on the hyperboloid. But,every point on the line PQ will also satisfy the conic and hence the conic contains a line. Since,the only conic that does so is a pair of straight lines, at most 2 lines pass through every point. We now only need to prove that no degenerate case occur i.e. pair of equal line.

- Case 1 $[z_0=x_0y_0=0]$
  Let,$P=(1,0,0)$.Therefore,
  $$\Rightarrow a=1,b^2-c^2=0$$
  So, we get a pair of straight lines namely,(1,t,t) and (1,t,-t).
- Case 2$[z_0=0,x_0y_0\neq 0]$
  $$ax_0+by_0=1$$
  $$\Rightarrow a=\frac{1-by_0}{x_0}$$

Substituting in the hyperboloid equation,

$$\frac{1-by_0}{x_0}^2 + b^2 = 1 + c^2$$
$$(1-by_0)^2 + x_0^2 b^2 = x_0^2 + x_0^2 c^2$$
$$b^2 - x_0^2 c^2 - 2y_0 b + y_0^2 = 0$$
$$(b-y_0)^2 - (x_0 c)^2 = 0$$

So we once again get 2 sets of lines.

- Case 3 [$z_0 \neq 0$]
  Let $L = (a+tv_1, b+tv_2, z_0)$ pass through P.

  $$\Rightarrow (a+tv_1)^2 + (b+tv_2)^2 = 1 + z_0^2 \; \forall\, t \in \mathbb{R}$$
  $$\Rightarrow v_1^2 + v_2^2 = 0$$

  Thus, any line through it cannot be parallel to the XY-plane and hence intersects it at say $(a,b,0)$

  $$\Rightarrow ax_0 + by_0 = 1$$

  Since,$(a,b,0)$ lies on both the unit circle in the XY-plane and the line defined by $xx_0 + y_0 - 1 = 0, z = 0$, there can be at most 2 such points.
  Perpendicular distance of the line $xx_0 + yy_0 - 1 = 0, z = 0$ from (0,0,0) is $\frac{1}{1+z_0^2} < 1$ Therefore,exactly 2 such points (and hence 2 lines through P) exist.

■

Theorem 2. *Let P_1 and P_2 be 2 distinct points.Let L_1, L_2 and L_3, L_4 be the 2 pair of lines through P_1 and P_2 respectively. Then each of L_3, L_4 can be obtained by rotating L_1, L_2 by some angle (not necessarily same) about the z-axis.*

Proof. Let L be a line through $P = ((1+z_0^2)^{\frac{1}{2}}\cos(\alpha), (1+z_0^2)^{\frac{1}{2}}\sin(\alpha), z_0)$ and $Q = (\cos(\beta), \sin(\beta), 0)$
The necessary and sufficient condition for L to lie on the hyperboloid is

$$(1+z_0^2)^{\frac{1}{2}}(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)) = 1$$
$$(1+z_0^2)^{\frac{1}{2}}\cos(\alpha - \beta) = 1$$

If L is rotated by an angle θ about the z-axis.,the points P and Q are transformed to

$$P = ((1+z_0^2)^{\frac{1}{2}}\cos(\alpha+\theta), (1+z_0^2)^{\frac{1}{2}}\sin(\alpha+\theta), z_0)$$
$$Q = (\cos(\beta+\theta), \sin(\beta+\theta), 0)$$

And the equation is still satisfied.

Let Q be a point on line L_1 such that z-coordinate of Q and P_2 is same. Then Q (along with L_1) can be rotated by some angle(say,θ) about the z-axis to coincide with P_2. Now 3 lines pass through P_2,L_3, L_4 and the rotated L_1. But we have already proved that exactly 2 lines pass through a point. Thus, one of L_3, L_4 is equal to the rotated L_1 . Also lines which can be transformed into one another by rotation are said to be in the same family. Thus, there are 2 families of lines on the hyperboloid.

■

Theorem 3. *Lines of different families are coplanar and same family are skew.*

- *Proof.* Let P_1 and P_2 be 2 distinct points.Let L_1, L_2 and L_3, L_4 be the 2 pair of lines through P_1 and P_2 respectively. Let L_1 and L_3 belong to the same family.

Say that L_4 intersects the plane $z = 0$ at $(a, b, 0)$ and L_1 at $(x, y, 0)$.

$$L_4 = (a + tw_1, b + tw_2, tw_3)$$
$$L_1 = (x + tv_1, y + tv_2, tv_3)$$

As,L_1 lies in the hyperboloid.

$$(x + tv_1)^2 + (y + tv_2)^2 = 1 + (tv_3)^2 \; \forall \, t \in R$$
$$\Rightarrow t^2(v_1^2 + v_2^2 - v_3^2) + 2t(xv_1 + yv_2) = 0$$
$$\Rightarrow v_1^2 + v_2^2 - v_3^2 = 0, xv_1 + yv_2 = 0$$

Assuming $y \neq 0$,

$$(v_1, v_2, v_3) = (v_1, \frac{-xv_1}{y}, \frac{\pm v_1}{y})$$

Similarly,

$$(w_1, w_2, w_3) = (w_1, \frac{-aw_1}{b}, \frac{\pm w_1}{b})$$

Since,L_1 and L_4 belong to different family, it can be checked that the vectors must have opposite signs in the z-component.

Now, L_1 and L_4 are coplanar iff

$$D = \begin{vmatrix} w_1 & w_2 & w_3 \\ a - x & b - y & 0 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0$$

$$D = (a - x)(w_3 v_2 - w_2 v_3) + (b - y)(w_1 v_3 - w_3 v_1)$$

Substituting the values for $v_i s$ and $w_i s$,

$$D = (a-x)(\frac{-w_1}{b}\frac{-xv_1}{y} - \frac{-aw_1}{b}\frac{v_1}{y}) + (b-y)(w_1\frac{v_1}{y}) - \frac{-w_1}{b}v_1)$$
$$D = \frac{v_1 w_1}{by}((a-x)(a+x)+(b-y)(b+y))$$
$$D = \frac{v_1 w_1}{by}(a^2-x^2+b^2-y^2) = 0$$

Therefore,lines of different families are coplanar.
Now consider L_1 and L_3, L_1 and L_3 don't intersect because if not, then their point of intersection P and its rotated point P' should lie on L_1 but the line passing through PP' is parallel to z-axis.
If$L_1 \parallel L_3$,then,certainly L_1 intersects with L_4 as they are coplanar and $L_3 \nparallel L_4$. Therefore,L_1 lies in the plane through L_3 and L_4.But this is impossible as the plane intersects the hyperboloid in exactly 2 lines. Hence,L_1 and L_3 are skew.

■

This gives us a correspondence between, {lines that intersect 3 skew lines} $\subsetneq$ {Points on a quadric} They are not equal in the Affine space as there exist points on the hyperboloid which gives lines that are parallel. If we however see things in the Projective space,we get a one-to-one correspondence as parallel lines meet at infinity. This provides us a motivation to study things in the projective space as results get tied up. Also,we have found a correspondence between the space of lines and a *projective variety*. The notion of varieties is a fundamental one in Algebraic geometry and it is a very useful to be able to able to view geometric objects as a variety in some space.This allows us to use all the developed algebraic geometry machinery to study them and gain a better insight.Let us first review some basic definitions and results that we will need before proceeding further.

3. Basic definitions

Definition 3.0.1 An *affine variety* is the locus where a collection of polynomial equations is satisfied, i.e., given $F = \{f_j\}_{j \in J} \subset k[x_1, \ldots, x_n]$ we define

$$V(F) = \{a \in \mathbb{A}^n(k) | f_j(a) = 0 \, \forall \, j \in J\} \subset \mathbb{A}^n(k)$$

R Note that some texts define what as algebraic set what we have defined here as variety and use the term variety for an irreducible algebraic set.

Definition 3.0.2 The algebro-geometric closure of a subset $S \subset \mathbb{A}^n(k)$ is defined

$$\bar{S} = \{a \in \mathbb{A}^n(k) | f(a) = 0 \, \forall \, f \in I(S)\} = V(I(S))$$

A subset $S \subset \mathbb{A}^n(k)$ is closed if $S = \bar{S}$; $U \subset \mathbb{A}^n(k)$ is open if its complement $\mathbb{A}^n(k) \setminus U$ is closed in $\mathbb{A}^n(k)$.

Definition 3.0.3 The projective space associated with a finite dimensional vector space V is defined by

$$\mathbb{P}(V) := V \setminus \{0\} / \sim$$

where the equivalence relation is given by

$$u \sim v \Rightarrow \exists \lambda \in k^* \text{such that } u = \lambda v$$

$\mathbb{P}(\mathbb{A}_k^{n+1})$ is denoted by $\mathbb{P}_k^n$ Also, The dimension of $\mathbb{P}(V)$ is defined by $dim\mathbb{P}(V) := dimV - 1$.

Definition 3.0.4 A projective variety is a subset $V \subset \mathbb{P}_k^n$ such that there is a set of homogeneous polynomials $T \subset k[x_0, \ldots, x_n]$ with

$$V = \{P \in \mathbb{P}_k^n | f(P) = 0 \, \forall \, f \in T\}$$

4. Space of all lines in $\mathbb{P}^n$

On the face of it, the space of all lines doesn't appear to be a variety. First let us look at some specific examples to get a better understanding and then build up on it to the general version.

4.1 Lines in $\mathbb{R}^2, \mathbb{P}^2$

The general equation of a line in the X-Y plane is

$$ax + by + c = 0$$

With the condition that $a, b \neq (0,0)$. Also line defined by the 3-tuple (a,b,c) is same as the one defined by$(\lambda a, \lambda b, \lambda c)$ This hints towards $\mathbb{P}^2$. So, let the mapping, ϕ : Lines in $\mathbb{R}^2 \longrightarrow \mathbb{P}^2$ $\phi(ax + by + c = 0) = (a : b : c)$ It is easy to see that ϕ is injective but not surjective as the point (0:0:1) in $\mathbb{P}^2$ has no pre-image. Again this can be remedied by working in $\mathbb{P}^3$ as then this would have a pre-image that is known as the *line at infinity*. So, $\{$ lines in $\mathbb{P}^2\} \cong \mathbb{P}^2$

4.2 Lines in $\mathbb{R}^3$

To define a line in $\mathbb{R}^3$ we need either 2 points or 1 point and a direction vector.But there is no unique representation as any point on the line and any scalar multiple of the direction vector gives us the same line. We can deal with scalar multiples by looking at it in the projective space as we have done above but handling the equivalence due to the point is trickier. The idea here is to eliminate the point and represent the line by 2 directions so that the only equivalence that remains is of scalar multiples which is easily taken care of.
So, let the line be defined by the point p and vector $\vec{v} \neq \vec{0}$.Take $\vec{p}$ as the position vector and define,

$$\vec{m} = \vec{p} \times \vec{v}$$

Note that if the line passes through the origin,then m is 0. We can verify that this is invariant with respect to choice the point.

$$\vec{m}' - \vec{m} = (\vec{p}' \times \vec{v}) - (\vec{p} \times \vec{v})$$
$$\vec{m}' - \vec{m} = (\vec{p}' - \vec{p}) \times \vec{v}$$
$$\vec{m} = \vec{m}' \Leftrightarrow \vec{p}' = \vec{p} + \lambda \vec{v}$$

Thus the line can be represented by $(\vec{v},\vec{m})$.These are known as the Plücker coordinates [1] of the line and '$\vec{m}$' is also called the *moment* of the line. Since$\vec{m}$ and $\vec{v}$ are vectors they are defined uniquely up to scalar multiplication. Hence the map ,

$$(\vec{v},\vec{m}) \rightarrow (v_0 : v_1 : v_2 : m_0 : m_1 : m_2) \subset \mathbb{P}^5$$

Also, $\vec{m}.\vec{v} = 0$. So, the variety we are looking for is $V = \{(v_0 : v_1 : v_2 : m_0 : m_1 : m_2) \mid \sum_{i=0}^{2} m_i v_i = 0\} \subset \mathbb{P}^5$. To define the inverse for any point in the variety such that $(v_0 : v_1 : v_2) \neq (0:0:0)$,define

$$\vec{p} = \frac{(\vec{v} \times \vec{m})}{\|v^2\|}$$
$$\vec{p} \times \vec{v} = (\frac{\vec{v} \times \vec{m}}{\|v^2\|}) \times \vec{v}$$
$$\vec{p} \times \vec{v} = \frac{\|v^2\|\vec{m} - (\vec{v}.\vec{m})\vec{v}}{\|v^2\|}$$
$$\vec{p} \times \vec{v} = \vec{m}$$

Therefore we get back a unique line. Thus, the space of lines in $\mathbb{R}^3$ is embedded in the variety but not equal to it. Hence, $\{\text{Space of lines in } \mathbb{R}^3\} \cong V \setminus \{(0:0:0:m_1:m_2:m_3)\}$ So the space of lines is isomorphic not to a variety but to a *quasi-projective variety* whose closure is V. The lines corresponding to the case when $(v_0 : v_1 : v_2) = (0:0:0)$ can be found then in the projective space and let us see how.

4.3 Lines in $\mathbb{P}^3$

In $\mathbb{P}^3$,the line is not represented by direction vectors and the also the idea of cross product breaks down. So, it seems that we may have to start all over again to define the Plücker coordinates of the line . But how do we get started?
Lines in $\mathbb{P}^3$ through the points $(x_0 : y_0 : z_0 : t_0), (x_1 : y_1 : z_1 : t_1)$ are represented as,

$$\{(ux_0 + vx_1 : uy_0 + vy_1 : uz_0 + vz_1 : uz_0 + vz_1) | (u,v) \in \mathbb{P}^1\}$$

[1]Named after the German mathematician and physicist Julius Plücker (1801 – 1868)

Let us first calculate the Plücker coordinates of the line through (x_0, y_0, z_0) , (x_1, y_1, z_1) in $\mathbb{R}^3$.

$$\vec{v} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$$
$$\vec{m} = (x_0, y_0, z_0) \times \vec{v}$$
$$\vec{m} = (y_0 z_1 - z_0 y_1, z_0 x_1 - x_0 z_1, x_0 y_1 - y_0 x_1)$$

Therefore, the Plücker coordinates are $(x_1 - x_0 : y_1 - y_0 : z_1 - z_0 : y_0 z_1 - z_0 y_1 : z_0 x_1 - x_0 z_1 : x_0 y_1 - y_0 x_1)$ From this,we get the idea that for a line in $\mathbb{P}^3$ the Plücker coordinates could be defined as

$$(w_0 x_1 - x_0 w_1 : w_0 y_1 - y_0 w_1 : w_0 z_1 - z_0 w_1 : y_0 z_1 - z_0 y_1 : z_0 x_1 - x_0 z_1 : x_0 y_1 - y_0 x_1)$$

These coordinates can also be viewed as the determinants of the 2×2 submatrix of the matrix

$$\begin{bmatrix} x_0 & y_0 & z_0 & w_0 \\ x_1 & y_1 & z_1 & w_1 \end{bmatrix}$$

Firstly , it can be easily verified that the map is homogeneous and for any 2 points on the line the corresponding coordinates are same.Thus,the map is well-defined. Also the map lies in $V = \{(v_0 : v_1 : v_2 : m_0 : m_1 : m_2) \mid \sum_{i=0}^{2} m_i v_i = 0\} \subset \mathbb{P}^5$

Theorem 4. *The mapping defined above is injective.*

Proof. For any line we can choose u,v such that $w_1 = 0$.
So without loss of generality,we will assume that the w-coordinate of the second point is 0.
Now let 2 pair of points $(P_0, P_1) = ((x_0 : y_0 : z_0 : t_0), (x_1 : y_1 : z_1 : 0))$ and
$(Q_0, Q_1) = ((a_0 : b_0 : c_0 : d_0), (a_1 : b_1 : c_1 : 0))$ define the same line.

$$(w_0 x_1 : w_0 y_1 : w_0 z_1 : y_0 z_1 - z_0 y_1 : z_0 x_1 - x_0 z_1 : x_0 y_1 - y_0 x_1) =$$

$$(d_0 a_1 : d_0 b_1 : d_0 c_1 : b_0 c_1 - c_0 b_1 : c_0 a_1 - a_0 c_1 : a_0 b_1 - b_0 a_1)$$

We can make the coordinates exactly equal by scaling the first point by a factor.
Case 1 $[w_0 = 0]$
This implies that $d_0 = 0$ since $(a_1, b_1, c_1) \neq (0,0,0)$ Now we can interpret each 3-tuple (the first 3 coordinates) (x_0, y_0, z_0) as a position vector in $\mathbb{R}^3$ So, the condition thus becomes,

$$\vec{P}_0 \times \vec{P}_1 = \vec{Q}_0 \times \vec{Q}_1$$

That is , the normal to the planes defined by the vectors $\vec{P}_0, \vec{P}_1$ is same as the one defined by $\vec{Q}_0, \vec{Q}_1$. Also both contain the origin (0,0,0).Thus , the 4 vectors are coplanar and hence each $\vec{Q}_i$ can be obtained by a linear combination of $\vec{P}_0, \vec{P}_1$. Thus, all 4 points lie on the same line.

Case 2 [$w_0 \neq 0$]

$$w_0 x_1 = d_0 a_1 \Rightarrow \frac{d_0}{w_0} = \frac{x_1}{a_1}$$

Similarly from the other 2 equations we get,

$$\frac{d_0}{w_0} = \frac{y_1}{b_1} = \frac{z_1}{c_1}$$

Hence,the points P_1 and Q_1 are same. Also as d_0 and w_0 are non-zero we can divide by them to make the last coordinate 1. Now we have, $P_0' = (x : y : z : 1)$ $P_1 = Q_1 = (a_1 : b_1 : c_1 : 0)$ and $Q_0' = (a : b : c : 1)$ We have to now prove that Q_0' lies on line defined by P_0', P_1 Again as before , form vectors out of the first 3 coordinates. Note that if $\vec{P_0'} = \vec{0}$ then $\vec{P_0'} = \vec{0}$ and hence the proof is complete. Else, we have 3 non-zero vectors such that

$$\vec{P_0'} \times \vec{P_1} = \vec{Q_0'} \times \vec{P_1}$$

$$(\vec{Q_0'} - \vec{P_0'}) \times \vec{P_1} = 0$$

$$(\vec{Q_0'} - \vec{P_0'}) = k\vec{P_1}$$

Hence, $\vec{Q_0'} = \vec{P_0'} + k\vec{P_1}$ Since, last coordinate of $P_0' = Q_0'$ and $P_1 = 0$, the Q_0' is just a linear combination of P_0' and P_1 and hence lies on the line.
Thus , the mapping is injective. ■

The reader may feel that the above given proof is unnecessarily long and a more concise proof is possible.This is indeed the case and such a proof will be given for the general case.This proof has been given here as I feel it gives a more intuitive picture.

Now let us try to find an inverse of the map.If we can show that there is well defined inverse for all elements in the variety then it proves that the map is bijective. Let $(a_1 : a_2 : a_3 : b_1 : b_2 : b_3)$ be a point in V. To define the inverse we again divide it into 2 cases.
Case 1 [$(a_1 : a_2 : a_3) = (0 : 0 : 0)$]
Then, at least one of the $b_i \neq 0$ say b_1.
Define inverse as $P_0 = (-b_2 : b_1 : 0 : 0), P_1 = (\frac{-b_3}{b_1} : 0 : 1 : 0)$. It can be checked that taking their image we get the point $(0 : 0 : 0 : b_1 : b_2 : b_3)$.

Case 2 [$(a_1 : a_2 : a_3) = (0 : 0 : 0)$]
Then as we have seen previously ,we can find a inverse line in $\mathbb{R}^3$. Take any 2 points on it define the inverse as $(P_0, P_1) = ((x_0 : y_0 : z_0 : 1), (x_1 : y_1 : z_1 : 1))$.
Thus we have found an inverse well defined on the entire variety. Hence, the map is a bijection. We can also see that the lines whose inverse was "missing“ in the case of the real space were those lines

whose last coordinate was always 0.These lines lie completely out of the real space and are the *lines at infinity*.
Thus, we can see how we get a more complete mapping in the projective case and gives us enough motivation to work in the projective space to avoid dealing with cases.

4.4 Lines in $\mathbb{P}^n$

We can very easily extend the above used idea of Plücker coordinates to n dimensions but let us first introduce a much clearer notation. Let $(a_{11} : a_{12} : \cdots : a_{1n})$ and $(a_{21} : a_{22} : \cdots : a_{2n})$ be 2 different points defining a line L in $\mathbb{P}^n$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$$

The Plücker coordinates are defined by

$$p_{ij} = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix} = a_{1i}a_{2j} - a_{1j}a_{2i}$$

Thus, the map $\phi(L)$ sits inside $\mathbb{P}^N, N = \binom{n+1}{2} - 1$. It is easy to see that the map is well- defined.Let us show that the map is injective.

Theorem 5. *ϕ is injective.*

Proof. Let L_1, L_2 be 2 lines that map to the same coordinates.Let P be the point of intersection (Remember, any two lines meet in projective space). Let Q_1, Q_2 be any 2 point apart from P on L_1, L_2 respectively.Then finding the coordinates with respect to these and equating,

$$P_i Q_{1j} - P_j Q_{1i} = P_i Q_{2j} - P_j Q_{2i}$$

$$P_i(Q_{1j} - Q_{2j}) = P_j(Q_{1i} - Q_{2i}) \,\forall\, i, j$$

Now at least one of the $P_i s$ are non-zero say P_0.

$$Q_{1j} - Q_{2j} = P_j \left(\frac{Q_{10} - Q_{20}}{P_0} \right) \,\forall\, j$$

$$\Rightarrow Q_{1j} = Q_{2j} + kP_j$$

Thus Q_1 lies on L_2 which is a contradiction. ■

It is no longer straightforward to write the equations for the image variety as we longer have the dot product intuition. However, we can make our task easier by looking at the image in each of the open affine cover of $\mathbb{P}^N$. So let $U_{i_0j_0}$ be the open cover defined by $p_{i_0j_0} \neq 0$.

$$\begin{vmatrix} a_{1i_0} & a_{1j_0} \\ a_{2i_0} & a_{2j_0} \end{vmatrix} \neq 0$$

Thus this matrix has an inverse.This can be used to do a Gaussian elimination which is basically a linear combination of the rows.Thus this does not affect the line but only changes the 2 representative points.

$$\begin{bmatrix} a'_{11} & a'_{12} & \cdots & 1 & \cdots & 0 & a'_{1n} \\ a'_{21} & a'_{22} & \cdots & 0 & \cdots & 1 & a'_{2n} \end{bmatrix}$$

Using this we get the following relations,

$$\Rightarrow p_{i_0j} = a'_{2j}$$

$$\Rightarrow p_{ij_0} = a'_{1i}$$

Hence ,

$$p_{ij} = \begin{cases} p_{ij_0}p_{i_0j} - p_{jj_0}p_{i_0i} & (i,j) \neq (i_0,j_0) \\ 1 & (i,j) = (i_0,j_0) \end{cases}$$

These equations define an affine variety $V_{ij} \subset U_{ij}$ and it can be easily seen that for every set of values of $\{a'1i, a'2j\}$ the p_{ij} are uniquely determined.Therefore , $V_{ij} \cong \mathbb{A}^{2n-2}$

Now Homogenizing the equation we get,

$$p_{ij}p_{i_0j_0} = p_{ij_0}p_{i_0j} - p_{jj_0}p_{i_0i}$$

Now we check whether this relation holds if $p_{i_0j_0} = 0$,

$$\Rightarrow \begin{vmatrix} a_{1i_0} & a_{1j_0} \\ a_{2i_0} & a_{2j_0} \end{vmatrix} = 0$$

$$p_{ij_0}p_{i_0j} = (a_{1i}a_{2j_0} - a_{2i}a_{1j_0})(a_{2j}a_{1i_0} - a1ja_{2i_0})$$

$$p_{jj_0}p_{i_0i} = (a_{1j}a_{2j_0} - a_{2j}a_{1j_0})(a_{2i}a_{1i_0} - a_{1i}a_{2i_0})$$

Atleast one of the a_{1i_0}, a_{1j_0} is non-zero (if not the relation is trivially satisfied),say a_{2j_0}. Now , substituting a_{1i_0} by $\frac{a_{1j_0}a_{2i_0}}{a_{2j_0}}$ we get the relation. Hence, this relation is true for the entire image of ϕ. Hence, this relation is a necessary condition for a point to lie in the image.

Taking the closure of these varieties in $\mathbb{P}^N$, we get a projective variety V defined by the ideal

$$J = (\{p_{ij}p_{kl} = p_{il}p_{kj} - p_{jl}p_{ki}\})$$

The inverse can be easily defined. Let $p \in V$ be given . Then at least one of them say $p_{i_0 j_0} \neq 0$. First divide the coordinates by $p_{i_0 j_0}$ and then map,

$$a'_{1i} = p_{i j_0}$$

$$a'_{2j} = p_{i_0 j}$$

Thus, it is also a sufficient condition. Hence, we shown that the image of ϕ is a projective variety. Also, a question arises that if all the lines in $\mathbb{P}^n$ can be given the structure of a projective variety, then can we do something similar with a general linear subspace of dimension k .

5. Grassmannian

The idea of Grassmannian[1] is to view the finite dimensional subspaces of a vector space as a projective variety. Thus, we have the following definition,

Theorem 6. *The Grassmannian $G(k,n)$ is defined as the set of all k-dimensional linear subspaces of an n-dimensional vector space $V \cong K^n$. Also written as $G(k,V)$ to signify lack of choice of basis.*

Another alternate definition is $\mathbb{G}(k,n)$ as set of k-planes in $\mathbb{P}^n$. Therefore, $\mathbb{G}(k-1,n-1) = G(k,n)$. It is clear from the definition that $G(1,n) = \mathbb{P}^n$ and we have already seen $G(2,3) = \mathbb{P}^2$ and have identified $G(2,n)$ with a variety in $\mathbb{P}^{\binom{n}{2}-1}$ .Define ,

$$N = \binom{n}{k} - 1$$

We will now show a bijective correspondence of $G(k,n)$ with a variety in $\mathbb{P}^N$ .The proof is very similar to the one for $G(2,n)$
First fix a basis for the vector space and then to a k-dimensional vector space spanned by the vectors $[v_1, v_2, .., v_k]$ associate a $k \times n$ matrix where i^{th} row represents the coordinates of v_i with respect to our basis.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kj} & \cdots & a_{kn} \end{bmatrix}$$

Now this matrix is not unique to a subspace as left-multiplying it with an invertible $k \times k$ matrix represents the same subspace as it corresponds to a basis change of the subspace. Note,however that the basis of the vector space is not altered.

[1]Named after the German polymath Hermann Günther Grassmann(1809 – 1877)

Let $M_{j_1 j_2 .. j_k}$ denote the $k \times k$ matrix obtained from the j_i numbered columns and let $p_{j_1 j_2 .. j_k}$ denote its determinant. Now the Plücker coordinates corresponding to the subspace are

$$\{p_{j_1 j_2 .. j_k} | 1 \leq j_1 < j_2 < ... j_k \leq n\}$$

The coordinates are well-defined up to scalar if a different basis is chosen for the subspace the the new matrix is obtained by left- multiplying with the transformation matrix T.Hence, the Plücker coordinates get multiplied by a factor equal to $det(T)$ .Also, all the coordinates can't be 0 as that would imply that the rank of the matrix $< k$ and hence that the basis are linearly dependent.

Theorem 7. *The mapping from the subspace to the coordinates is injective.*

Proof. Let $[v_1, v_2, .., v_k]$ and $[w_1, w_2, .., w_k]$ span 2 k-dimensional vector spaces V and W respectively,such that their Plücker coordinates are the same.Let $w^* \in W$ Now, from linear algebra we know that $w^* \in W$ iff

$$rank \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1j} & \cdots & w_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{i1} & w_{i2} & \cdots & w_{ij} & \cdots & w_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{k1} & w_{k2} & \cdots & w_{kj} & \cdots & w_{kn} \\ w_1^* & w_2^* & \cdots & w_j^* & \cdots & w_n^* \end{bmatrix} = k$$

This means that the determinant of all $k+1 \times k+1$ matrices are 0. Expanding these along w_i^* we get set of linear relations with the coordinates as the coefficients. But, the coordinates are same even if we replace the w_{ij} by v_{ij}.Hence,

$$rank \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1j} & \cdots & v_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{i1} & v_{i2} & \cdots & v_{ij} & \cdots & v_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{kj} & \cdots & v_{kn} \\ w_1^* & w_2^* & \cdots & w_j^* & \cdots & w_n^* \end{bmatrix} = k$$

$$\therefore w^* \in V$$

■

Now, like we did earlier, we try to find the equations that define the variety corresponding to the image of the map by looking at its intersection with the affine covers over $\mathbb{P}^N$. So let the open sets $U_{j_1 .. j_k}$ be defined by $p_{j_1 .. j_k} \neq 0$. So let us look at $U_{12...k}$ just for the sake of notational simplicity.(Just replacing $1, 2, .., k$ by $j_1, j_2, .. j_k$ in the entire argument will make it "general").

As $p_{1..k} \neq 0$, the matrix $M_{1..k}$ has an inverse and this when multiplied to the coefficient matrix will lead to a change of basis such that the matrix defined first k rows and columns becomes identity.

$$\left[\begin{array}{ccc|ccc} 1 & \cdots & 0 & v'_{1k+1} & \cdots & v'_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & v'_{kk+1} & \cdots & v'_{kn} \end{array}\right]$$

It can be clearly seen that,

$$p_{1..\hat{i}j..k} = v'_{ij} \quad j > k \tag{5.1}$$

Note - $\hat{i}$ means absence of i^{th} column

And as all $p_{j_1..j_k}$ are polynomials in v'_{ij}, we get a set of polynomial relations in the coordinates and hence define an *affine variety*. We also see that the $k(n-k)$ coordinates of the form $p_{1..\hat{i}j..k}$ uniquely determine all the other coordinates and hence this variety is isomorphic to $\mathbb{A}^{k(n-k)}$.Thus, the Grassmannian's dimension is $k(n-k)$.

The equations that we get by the above method has polynomials of deg(k) and it is almost always desirable to get equivalent set of equations with a lesser degree.In our case we can reduce the degree to 2.Let us see how.

$$p_{j_1 j_2..j_k} = \begin{vmatrix} v_{1j_1} & v_{1j_2} & \cdots & v_{1j_r} & \cdots & v_{1j_k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{ij_1} & v_{ij_2} & \cdots & v_{ij_r} & \cdots & v_{ij_k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{kj_1} & v_{kj_2} & \cdots & v_{kj_r} & \cdots & v_{kj_k} \end{vmatrix}$$

Expanding with respect to the first column (assuming $j_1 > k$, else pick any other),

$$p_{j_1 j_2..j_k} = \sum_{t=1}^{k} (-1)^{t-1} v_{tj_1} \begin{vmatrix} v_{1j_2} & \cdots & v_{1j_r} & \cdots & v_{1j_k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{\hat{t}j_2} & \cdots & v_{\hat{t}j_r} & \cdots & v_{\hat{t}j_k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{kj_2} & \cdots & v_{kj_r} & \cdots & v_{kj_k} \end{vmatrix}$$

$$p_{j_1 j_2..j_k} = \sum_{t=1}^{k} (-1)^{t-1} v_{tj_1} \begin{vmatrix} 0 & v_{1j_2} & \cdots & v_{1j_r} & \cdots & v_{1j_k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & v_{tj_2} & \cdots & v_{tj_r} & \cdots & v_{tj_k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & v_{kj_2} & \cdots & v_{kj_r} & \cdots & v_{kj_k} \end{vmatrix}$$

$$\Rightarrow p_{j_1 j_2..j_k} = \sum_{t=1}^{k} (-1)^{t-1} v_{t j_1} p_{t j_2..j_k}$$
$$\Rightarrow p_{j_1 j_2..j_k} = \sum_{t=1}^{k} (-1)^{t-1} p_{12..\hat{t} j_1,..k} p_{t j_2..j_k}$$

Homogenizing,we obtain,

$$\Rightarrow p_{j_1 j_2..j_k} p_{12...k} = \sum_{t=1}^{k} (-1)^{t-1} p_{12..\hat{t} j_1,..k} p_{t j_2..j_k}$$

This equation also holds when $p_{12...k} = 0$.

Proof. As $p_{12...k} = 0$, the columns are not linearly independent and hence a relation,

$$\sum_{i=1}^{m} \lambda_i c_i = 0 \;\; c_i \in \{1, 2...k\}$$

The terms corresponding to $t \neq c_i$ are 0 as $M_{12..\hat{t} j_1,..k}$ will contain all the c_is and hence $p_{12..\hat{t} j_1,..k} = 0$ So the relation to prove becomes,

$$\sum_{i=1}^{m} (-1)^{c_i - 1} p_{12..\hat{c_i} j_1,..k} p_{c_i j_2..j_k} = 0$$

Also if $j_2, j_3, ..j_k$ are dependent then every term is 0 and hence it is trivial. So as they are independent we extend them by a basis v(in the k-dim vector space) .
Now decompose the column vector c_i in terms of $\{v, j_2, j_3...j_k\}$,

$$c_i = \alpha_{i1} v + \sum_{r=2}^{k} \alpha_{ir} j_r$$

Using the linear relation between the c_i we get,

$$\sum_{i=1}^{m} \alpha_{i1} \lambda_i = 0$$

Define ,

$$p* = det(M_{v j_2 j_3..j_k})$$

Hence,

$$p_{c_i j_2..j_k} = \alpha_{i1} p*$$

Also doing some elementary operations on the matrix we can convert all the $M_{12..\hat{c_i} j_1,..k}$ to the form

$p_{12..\hat{c}_1 j_1,..k}$. Thus,

$$\begin{aligned}
p_{12..\hat{c}_i j_1,..k} &= (-1)^{c_i - c_1} \frac{\lambda_i}{\lambda_1} p_{12..\hat{c}_1 j_1..k} \\
\sum_{i=1}^{m} (-1)^{c_i - 1} p_{12..\hat{c}_i j_1,..k} p_{c_i j_2..j_k} &= \sum_{i=1}^{m} (-1)^{c_1+1} \frac{\lambda_i}{\lambda_1} p_{12..\hat{c}_1 j_1,..k} \alpha_{i1} p* \\
&= (-1)^{c_1+1} \frac{p_{12..\hat{c}_1 j_1,..k} p*}{\lambda_1} \sum_{i=1}^{m} \lambda_i \alpha_{i1} \\
&= 0
\end{aligned}$$

■

Hence the equations can be generalized to the form,

$$\Rightarrow p_{j_1 j_2..j_k} p_{c_1 c_2...c_k} = \sum_{t=1}^{k} (-1)^{t-1} p_{c_1 c_2..\hat{c}_t j_1,..k} p_{c_t j_2..j_k}$$

And these are a set of necessary relation of degree 2 that every element should satisfy for it to lie in the image. They are also sufficient as given a set of coordinates satisfying the above relations,we can construct the image by setting , (Assuming $p_{c_1 c_2..c_k} = 1$)

$$\begin{aligned}
&v_{ij} = p_{c_1..\hat{c}_i j..c_k} \;\; j > k \\
&v_{jj} = 1 \;\; j \leq k \\
&v_{ij} = 0 \;\; j \leq k \; i \neq j
\end{aligned}$$

Thus we have proved that there is bijection between the Grassmannian $G(k,n)$ and a quadric in $\mathbb{P}^N$.

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